# Math 128, Modern Geometry

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16 Sep 2005

**Due Today.** Exercises from chapter 4: exercises 1, 2, 5.

**Today.** Begin Möbius geometry.

**The definition of Möbius geometry.** We'll describe this geometry via a transformation group on a set, then investigate its invariants. The underlying space of Möbius geometry is $\mathbf{C}^+$, the complex plane with one element, the point at $\infty$, added. A transformation $T : \mathbf{C}^+ \to \mathbf{C}^+$ of this geometry is of the form

$$T(z) = \frac{az + b}{cz + d}$$

where $a$, $b$, $c$, and $d$ are complex constants such that $ad - bc \neq 0$. You probably recognize $ad - bc$ as the determinant of a $2 \times 2$ matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. There's a way of analyzing all the geometry we're doing in terms of matrices and linear algebra, but we won't be looking at that. Such a transformation is called a *Möbius transformation* or a *fractional linear transformation*.

These Möbius transformations do in fact form a transformation group $G$ becase they have the three properties required to be a transformation group.

First, the identity transformation $I$ is included in $G$. That's the transformation $I$ defined by $I(z) = z = \frac{1z+0}{0z+1}$ is a Möbius transformation.

Second, for each transformation $T \in G$, there is an inverse transformation $T^{-1} \in G$ because

$$w = T(z) = \frac{az + b}{cz + d}$$

if and only if

$$z = T^{-1}(w) = \frac{dw - b}{-cw + a}.$$

Third, the composition $T \circ U$ of two Möbius transformations is another one. (See the text for details.)

**Geometric interpretation of Möbius transformations.** These Möbius transformations include several of the transformations we've looked at so far. All translations $T(z) = z + b = \frac{1z+b}{0z+1}$ are Möbius transformations. Also, all rotations about the origin $T(z) = e^{i\theta} z$ and scalings fixing the origin $T(z) = rz$ (with $r > 0$) are Möbius transformations since these are both of the form $T(z) = \frac{az+0}{0z+1}$. And, since Möbius transformations are closed under composition, that implies rotations and scalings about any point are Möbius transformations. Furthermore, reciprocation $T(z) = \frac{1}{z} = \frac{0z+1}{1z+0}$ is a Möbius transformation.

But reflections are not Möbius transformations. In particular, complex conjugation is not a Möbius transformation.

On the other hand, every Möbius transformation can be decribed as a composition of translations, rotation, scalings, and reciprocation. When $c \neq 0$,

$$T(z) = \frac{az + b}{cz + d} = \frac{1}{c} - \frac{ad - bc}{c^2} \left( z + \frac{d}{c} \right)^{-1},$$

which illustrates $T$ as a composition of these various operations. Likewise, when $c = 0$,

$$T(z) = \frac{az + b}{d} = \left( \frac{a}{d} \right) z + \left( \frac{b}{d} \right)$$

shows $T$ is also a composition of these various operations.