Math 128, Modern Geometry

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Due Friday. Exercises from chapter 4: exercises 1, 2, 5.

Read for Wednesday. Continue reading chapter 4.

Last time. Studied the stereographic projection. Discussed the purpose of Klein's Erlanger program.

Today. Klein's Erlanger Program. Klein's idea was to base a geometry, whichever one of these is under consideration, by taking as a basis the group of transformations of that geometry that preserves the properties of that geometry. For instance, if the geometry is the usual Euclidean plane geometry, then transformations of the plane that preserve distance are the appropriate transformations. That's enough, because if a tranformation preserves distance, then it preserves straight lines, parallelism, angles, areas, and any other Euclidean concepts. A transformation that preserves distance is called a *rigid motion* or an *isometry.*

Transformation groups. Defintion of a transformation group G on a set S. The points of the geometric space are the elements of the set S, sometimes called *the underlying space of the geometry.* Typically, these points are given coordinates, but the coordinates are for our convenience, they aren't intrinsic to the geometry. The group G is a group of transformations of S. That is, each element T of G is a transformation $T : S \to S$. That means, precisely, that T is a function that assigns to each point $x \in S$ another point $T(x) \in S$. We'll frequently leave out the parenthese and simply write Tx. Three requirements are placed on G in order for it to be a *group* of transformations.

First, the identity transformation I is included in G. That's the function I defined by $I(x) = x$ for all $x \in S$.

Second, for each transformation $T \in G$, there is an inverse transformation $U \in G$. For U to be an inverse of T means that $T(x) = y$ if and only if $x = U(y)$. The inverse transformation of T is usually denoted T^{-1}.

Third, if T and U are transformations in G, then their composition $T \circ U$ is also a transformation in G. The composition $T \circ U$ is defined as usual by $(T \circ U)(x) = T(U(x))$. We'll usually write the compostion without the circle, TU.

Klein's idea was to take a geometry as that which is completely determined by a transformation group G. It's clear the the points of the geometry are the elements of the underlying space S. But all the other geometric concepts are to be drawn from the group G. Thus, we can identify a geometry with (S, G), the pair consisting of a set S and a group of transformations G on that set.

Congruent figures in a geometry. First, we'll take as a figure any subset of the underlying space S. Subsets include the usual figures of study like triangles and circles, but they include lots of uninteresting things too. We'll include them all as being figures.

We'll say that two figures A and B are *congruent* in a geometry if there is some transformation T in the group G that sends A to B, that is $T(A) = B$, and by $T(A)$ we mean the figure of all points of the form $T(a)$ where a ranges over the figure A,

$$T(A) = \{T(a) : a \in A\}.$$

When A and B are congruent figures, we'll denote that as $A \cong B$.

Congruence as an equivalence relation. Note that congruence is an equivalence relation because it satisfies the three conditions to be an equivalence relation, namely, reflexivity, symmetry, and transitivity. These three properties correspond exactly to the three conditions necessary for G to be a group.

Congruence is reflexive since each figure A is congruent to itself, $A \cong A$. That's because the identity transformation I sends A to itself, that is, $I(A) = A$.

Congruence is symmetric since if A is congruent to B, then B is congruent to A. Why? Well, if $A \cong B$, then there is some transformation T such that $T(A) = B$. Then $A = T^{-1}(B)$, so $B \cong A$.

Congruence is transitive since if $A \cong B$ and $B \cong C$, then $A \cong C$. Here's the reason. If $A \cong B$, then $T(A) = B$ for some tranformation T. And if $B \cong C$, then $U(B) = C$ for some tranformation U. Then $UT(A) = U(B) = C$, so the composition UT sends A to C. Therefore, $A \cong C$.

Invariants of a geometry. We'll define two related concepts of invariance.

First, we'll say a set D of figures is an *invariant set of figures* if it's closed under congruence. That means if A is a figure in D, and $A \cong B$, then B is also a figure in D. In other words, if A is any figures in D, and T is any tranformation in the group G, then $G(A)$ is another of the figures in D.

Second, we'll say a function f defined on an invariant set D is an *invariant function* if it has the same value for all congruent figures. Equivalently, if A is any figure in D and T is a tranformation in G, then $f(T(A)) = f(A)$.

Examples. We'll look at some example plane geometries. We'll describe the sets S in terms of complex numbers $\mathbf{C}$, and the transformations as functions on the complex numbers. Our first example will be the usual Euclidean plane. Next, we'll consider a subgeometry of that which only has translations, called *translational geometry*. It's not particularly important, but it does give us an example of another geometry. We won't get to the more interesting geometries until later.