Math 128, Modern Geometry

D. Joyce, Clark University

2 Sep 2005

Due Today. More exercises from chapter 2: 9, 14a–c,e–f, 21a, 22.

Read for next Wednesday the rest of chapter 3.

Due next Friday. Exercises from chapter 3: exercises 1, 2, 5b, 6, 8, 10, 15.

Today. Discuss homework due. Study geometric transformations. Continue our investigation on the assumptions behind the Pythagorean theorem.

Introduction to geometric transformations. We're considering the usual Euclidean plane, and we'll coordinatize it with the complex numbers $\mathbf{C}$, so we'll often refer to 'the plane $\mathbf{C}$.'

A transformation f of the plance $\mathbf{C}$ is just a function $f : \mathbf{C} \to \mathbf{C}$. It associates to a point $z \in \mathbf{C}$ another point $w = f(z) \in \mathbf{C}$. We're not interested in alll possible transformations, just certain ones that are geometrically important. We'll look at a few of them today. We can find illustrations of some of them at the Wallpaper Groups web page `http://www.clarku.edu/~djoyce/wallpaper/`

Translations. A *translation* is a transformation $f : \mathbf{C} \to \mathbf{C}$ of the form $f(z) = z + b$ where b is a constant complex number, called the *displacement* of the translation.

The identity transformation. The identity transformation leaves every point fixed. It's the transformation $f(z) = z$. You can think of the identity as a special translation where the displacement $b = 0$. Sometimes, the identity transformation is called the *trivial transformation*.

Rotations. A *rotation* about the origin is of the form $f(z) = e^{i\theta} z$ where θ is a real constant, the angle of rotation. Note that if $\theta = 0$, then the rotation is the identity transformation.

Some rotations are of special interest. A *half-turn* is a rotation where the angle is π, that is, a 180°-rotation. A *quarter-turn* is a rotation where the angle is $\pi/2$, that is, a 90°-rotation.

We'll also look at rotations about points other than the origin.

Scalings. These are also called homothetic transformations, dilatations, etc. They include contractions and expansions with one fixed point. A scaling where the fixed point is the origin is of the form $f(z) = kz$ where k is a positive real constant called the *scaling factor*. If $k > 1$, then it's an expansion, but if $k < 1$, it's a constraction. And, of course, if $k = 1$, then it's the identity transformation.

We'll also look at scalings where the fixed point is a point other than the origin.

Reflections. In general, the word reflection, when applied to plane geometry, means reflection across a line. An example of this is complex conjugation $f(z) = \overline{z}$. That transformation reflects points across the real axis.

Inversion. This inversion is actually reciprocation, $f(z) = 1/z$. We'll look later at inversions in a circle, but this one is a little different from inversions in a circle.

Reciprocation is not defined for the point $z = 0$, but we can add one point to the plane, called *the point at infinity*, denoted ∞, so that reciprocation is defined everywhere. Then $1/0 = \infty$, and $1/infty = 0$. Inversion sends straight lines to circles passing through the origin, and vice versa. We'll see later that reciprocation sends any circle not passing through the origin to some other circle not passing through the origin, but that result takes a bit of work to prove.

Note that the composition of reciprocation with itself gives the identity transformation. Any transformation of order 2, that is, transformation whose composition with itself is the identity, is called an *involution*. Other examples of involutions are half-turns and reflections