# Math 128, Modern Geometry

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**Due Monday.** From chapter 13, exercises 1, 2, 6, 11.

**Last time.** Continued our introduction to projective geometry. We've seen three models for the projective plane.

(1) points of the projective plane are modelled by lines in $\mathbf{R}^3$ through the origin, while lines in the projective plane are modelled by planes in $\mathbf{R}^3$ through the origin.

(2) points of the projective plane are modelled by pairs of antipodal points on the unit sphere, while lines in the projective plane are modelled by great circles on the unit sphere.

(e) points of the projective plane are modelled by points in one hemisphere of the unit sphere (including half of the boundary of that hemisphere), while lines in the projective plane are modelled by the parts of great circles that lie in that hemisphere.

We also studied homogeneous coordinates $(x, y, z)$ for points and homogeneous coordinates $[a, b, c]$ for lines, and noted that a point $(x, y, z)$ lies on a line $[a, b, c]$ if an only if their dot product is 0, that is $[a, b, c] \cdot (x, y, z) = ax + by + cz = 0$. Since that equation is symmetric with respect to points and lines, therefore any theorem about points and lines in the projective plane yields a dual theorem where the role of lines and points are interchanged.

**Today.** Cross products. Projective transformations. The fundamental theorem for the projective plane.

**Cross products.** The cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ is another another vector denoted $\mathbf{u} \times \mathbf{v}$. The easiest way to define cross products is to use the standard unit vectors $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$. Then we can write

$$\mathbf{u} = (u_1, u_2, u_3) = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k},$$

and

$$\mathbf{v} = (v_1, v_2, v_3) = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k},$$

and $\mathbf{u} \times \mathbf{v}$ is defined as

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} + (u_3v_1 - u_1v_3)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

which is much easier to remember when you write it as a determinant

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \mathbf{k} \\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \end{aligned}$$

We can use cross products to work with points and lines in the projective plane. In particular, they can be used to find the line determined by two points, and to find the point determined by two lines.

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are the homogeneous coordinates for two points, then $\mathbf{u}\times\mathbf{v}$ gives the homogeneous coordinates for the line that passes through them. Dually, if $\mathbf{u} = [u_1, u_2, u_3]$ and $\mathbf{v} = [v_1, v_2, v_3]$ are the homogeneous coordinates for two lines, then $\mathbf{u} \times \mathbf{v}$ gives the homogeneous coordinates for the point where they intersect.

**Projective transformations.** We would like to put our study of projective geometry into Klein's Erlanger Programm. That means we need to know the group of transformations of the projective plane. In other words, we have to answer the question: what is a projective transformation. The answer is any transformation that preserves points and lines, or preserves collinearity. More precisely, it's a transformation $T$ on the set of points of the projective plane, and we'll model points using homogeneous coordinates, so that if $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are three collinear points, then so are $T(\mathbf{u})$, $T(\mathbf{u})$, and $T(\mathbf{u})$ collinear.

We can model such a transformation as an invertible function $T : \mathbf{R}^3 \to \mathbf{R}^3$ that sends $\mathbf{0}$ to $\mathbf{0}$, straight lines through $\mathbf{0}$ to straight lines through $\mathbf{0}$, and planes through $\mathbf{0}$ to planes through $\mathbf{0}$. These conditions are equivalent to requiring that $T$ be an invertible linear transformation. A transformation is *linear* if it preserves linear combinations, that is,

$$T(a\mathbf{u} + b\mathbf{v}) = aT(\mathbf{u}) + bT(\mathbf{v})$$

where $a$ and $b$ are any real numbers and $\mathbf{u}$ and $\mathbf{v}$ are any vectors.

A linear transformation $T$ can be described by a $3 \times 3$ matrix

$$T(\mathbf{u}) = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ dx + ey + fz \\ gx + hy + iz \end{bmatrix}$$

where $\mathbf{u}$ is the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and the matrix entries $a, \ldots, i$ are real constants.

**The fundamental theorem for the projective plane.** Recall the fundamental theorem for the Möbius plane. It said that there is a unique Möbius transformation that sends any given three points to any other given three points. The fundamental theorem for the projective plane is similar, but for four noncollinear points. If $\mathbf{p}_0$, $\mathbf{p}_1$, $\mathbf{p}_3$, and $\mathbf{p}_4$ are four points in the projective plane, no three of which are collinear, and $\mathbf{q}_0$, $\mathbf{q}_1$, $\mathbf{q}_3$, and $\mathbf{q}_4$ are four more points in the projective plane, no three of which are collinear, then there is a unique projective transformation that maps each point $\mathbf{p}_i$ to the corresponding point $\mathbf{q}_i$ for $i = 1, 2, 3, 4$.