Math 128, Modern Geometry

D. Joyce, Clark University

19 Oct 2005

Due Today. From Chapter 7: 1, 6, 9, 10.

Due Monday. From Chapter 8: 1, 2, 8.

Last time. Geometric classification of transformations of the hyperbolic plane. We found that besides the identity transformation, the transformations of the hyperbolic plane are

hyperbolic translations along one line in the plane that move points along hypercycles,

hyperbolic rotations around a point that move points along circles, or

parallel displacements that move points along horocycles.

Today. Chapter 9 on hyperbolic length. We'll start with two formulas for length. The first is a formula for the particular distance from 0 to a real number r between 0 and 1:

$$d(0,r) = \ln\left(\frac{1+r}{1-r}\right).$$

Note that as r approaches 1, the denominator $1-r$ approaches 0, so $\dfrac{1+r}{1-r}$ approaches ∞. Therefore $d(0,r)$ also approaches ∞. That means that the radius from 0 to 1 of the Poincaré disk $\mathbf{D}$ has infinite length as a ray in the hyperbolic plane.

The second formula for distance is the general one that gives the distance between two points z_1 and z_2 in $\mathbf{D}$.

$$d(z_1,z_2) = \ln\left(\frac{1+r}{1-r}\right) \text{ where } r = \left|\frac{z_2-z_1}{1-z_1\overline{z}_2}\right|.$$

That formula follows from the first after a transformation of the hyperbolic plane that maps $z_1 \mapsto 0$ and $z_2 \mapsto r$.

Properties of distance. The important thing about this distance is that all the transformations of the hyperbolic plane preserve it. That is, if T is any transformations of the hyperbolic plane and z_1 and z_2 any two points in the hyperbolic plane, then

$$d(z_1,z_2) = d(T(z_1),T(z_2)).$$

In other words, this distance is an invariant of hyperbolic geometry.

Many of the properties of distance in the Euclidean plane are shared by distance in the hyperbolic plane. First of all, distance is an equivalence relation on pairs of points in the hyperbolic plane. Of course, that means three things:

Reflexivity: $d(z_1,z_2) = d(z_1,z_2)$.

Symmetry: $d(z_1,z_2) = d(z_3,z_4)$ implies $d(z_3,z_4) = d(z_1,z_2)$.

Transitivity: If $d(z_1,z_2) = d(z_3,z_4)$ and $d(z_3,z_4) = d(z_5,z_6)$, then $d(z_1,z_2) = d(z_5,z_6)$.

Next, the distance between two points doesn't depend on the order the two points are named, it's nonnegative, and it's 0 if an only if the two points are the same.

$d(z_1,z_2) = d(z_2,z_1)$.

$d(z_1,z_2) \geq 0$.

If $d(z_1,z_2) = 0$, then $z_1 = z_2$.

So far, we haven't mentioned anything particularly geometric about distance. But there are important geometric properties, too. Here are three of them.

If z_1, z_2, and z_3 lie on a hyperbolic straight line in that order, then

$$d(z_1,z_3) = d(z_1,z_2) + d(z_2,z_3).$$

The shortest curve connecting two points z_1 and z_2 in the hyperbolic plane is the hyperbolic straight line joining them.

The triangle inequality: The sum of the lengths of two sides of a triangle is less than or equal the the length of the third. Symbolically,

$$d(z_1, z_3) \leq d(z_1, z_2) + d(z_2, z_3).$$