Math 128, Modern Geometry

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Due Wednesday. From Chapter 7: 1, 6, 9, 10.

For next time. Begin reading chapter 8.

Last time. Examined the beginning of Book I of Euclid's *Elements*. Before that, discussed Poincaré's model for hyperbolic geometry.

Summary of Poincaré's model for the hyperbolic plane.

The underlying space is the open unit disk

$$\mathbf{D} = \{z \in \mathbf{C} \mid |z| < 1\}$$

whose boundary is the unit circle

$$\partial\mathbf{D}\{z \in \mathbf{C} \mid |z| = 1\}$$

The points on the boundary do not belong to the hyperbolic plane, but they can be thought of as "points at ∞," and we'll call them *ideal points.*

The transformation group $\mathbf{H}$ on this space, the *hyperbolic group*, is the group of all Möbius transformations which map the unit disk $\mathbf{D}$ onto itself. Such a transformation is of the form

$$T(z) = e^{i\theta} \frac{z - z_0}{1 - \overline{z}_0 z}$$

where θ is an angle and z_0 is a complex constant $|z_0| < 1$, in fact, T maps the point z_0 to 0.

A straight line in this model of the hyperbolic plane as the part of a cline inside the unit disk that intersects the unit circle orthogonally.

So, for instance, the diameters of the unit disk are all hyperbolic straight lines, but there are a lot of other hyperbolic straight lines, too. The rest are all arcs of circles that meet the unit circle at right angles. Given any two points z and w in the hyperbolic plane, there is a unique hyperbolic straight line A passing through them.

Parallels in the hyperbolic plane.

Note that our textbook uses unusual terms for parallel lines. Hyperbolic lines that don't meet are ususally called *parallel* lines, but our text calls them *hyperparallel* lines. Hyperbolic lines that meet at ideal points are usually called *asymptotoically* parallel, but our text calls them *parallel.*

Hyperbolic geometry satisfies all the axioms for Euclidean geometry except one, namely the Parallel Postulate. In the hyperbolic plane, given a line and a point not on the line, there are infinitely many lines parallel to the given line passing through the point. In fact, there are exactly two asymptotically parallel lines, one asymtotically parallel to one end of the given line, the other asymptotically parallel to the other end of the given line.

Cycles in the hyperbolic plane.

So far, we've seen that the lines in the hyperbolic plane are arcs of clines in Möbius geometry that are orthogonal to ∂D. What about the other clines of Möbius geometry? What are they?

They come in three types depending on whether they're (1) entirely inside D, in which case we'll call them *hyperbolic circles*, (2) are inside D except where they touch ∂D at one point, in which case we'll call them *horocycles*, or (3) partly inside and partly outside D so that they meet ∂D at two points. In the third case, when the cline is not orthogonal to ∂D, we'll call the arc inside D a hypercycle. All three of these curves are interesting. The circles in (1) will actually be circles in the hyperbolic plane, but we haven't described their centers yet, since we haven't even got a concept of distance yet for the hyperbolic plane.