Math 128, Modern Geometry

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Due Friday. From Chapter 6: 3abc, 6, 7ab, 10.

Read Chapter 7. We'll start it today and finish it next time.

Poincaré's model for hyperbolic geometry. Rather than build hyperbolic plane geometry from axioms, we'll use a transformation group. Our underlying space is the interior of the unit circle in the complex numbers. That's called the (*open*) *unit disk*, denoted here $\mathbf{D}$.

$$\mathbf{D} = \{z \in \mathbf{C} \,|\, |z| < 1\}$$

The transformation group $\mathbf{H}$ on this space, called the *hyperbolic group*, is the group of all Möbius transformations which map the unit disk $\mathbf{D}$ onto itself. A little algebra, as seen in the text, shows that such a transformation is of the form

$$T(z) = e^{i\theta}\frac{z - z_0}{1 - \overline{z}_0 z}$$

where θ is an angle and z_0 is a complex constant $|z_0| < 1$.

Since the hyperbolic group $\mathbf{H}$ is a subgroup of the Möbius group, hyperbolic geometry is a subgeometry of Möbius geometry. That means hyperbolic geometry inherits the invariants of Möbius geometry. But it can also have invariants that aren't inherited from Möbius geometry.

Furthermore, figures that are congruent in Möbius geometry need not be congruent in hyperbolic geometry. That's because one figure can be mapped to another via a Möbius transformation that's not a hyperbolic transformation, that is to say, a Möbius transformation that doesn't map the unit disk to itself.

Definition of straight line in Poincaré's model. We define a *straight line* in this model of the hyperbolic plane as the part of a cline inside the unit disk that intersects the unit circle orthogonally.

So, for instance, the diameters of the unit disk are all hyperbolic straight lines, but there are a lot of other hyperbolic straight lines, too. The rest are all arcs of circles that meet the unit circle at right angles.

As described in the text, the set of all these hyperbolic straight lines is an invariant set of figures in hyperbolic geometry. That is, if A is a hyperbolic straight line, and T is a hyperbolic transformation, then $T(A)$ is another hyperbolic straight line. Furthermore, given any two hyperbolic straight lines A and B, there is a hyperbolic transformation T that maps $A \mapsto B$.

A related theorem states that given any two points z and w in the hyperbolic plane, there is a unique hyperbolic straight line A passing through them.