Math 128, Modern Geometry

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First test. This Friday, to cover through chapter 5. Review on Wednesday.

Today. We'll begin chapter 6 which looks into Möbius geometry in a more depth. We'll see what Steiner circles are. Actually, Steiner circles aren't circles, but families or sets of circles. They're also called pencils or bundles of circles. We'll use these Steiner circles to understand Möbius transformations better.

Families of Steiner circles of the first and second kinds. If p and q are two distinct points in $\mathbf{C}^+$, the set of all the clines (circles or lines) in $\mathbf{C}^+$ that pass through p and q form a family of *Steiner circles of the first kind*.

Note that one of the two points p or q may be the point at ∞, in which case all the clines are actually straight lines. Indeed, in order to study a family of Steiner circles better, we'll apply the Möbius transformation $S(z) = \dfrac{z-p}{z-q}$ which maps $p \mapsto 0$ and $q \mapsto \infty$. Then the family of Steiner circles is just the set of all straight lines through the origin, 0.

Two circles or lines are said to be *orthogonal* or *perpendicular* to each other if their tangent lines are at right angles. (It's funny that we have so many words to say two things are at right angles to each other. Besides saying that they're at right angles (English), we can say they're perpendicular (Latin), or orthogonal (Greek). Sometimes we say they're "normal." One term would be enough. Sigh.)

A family of *Steiner circles of the second kind* is the family of curves orthogonal to a family of Steiner circles of the first kind. To visualize it better, take a family of Steiner circles of the first kind, apply the transformation mentioned above so that it's transformed into the family of straight lines through the origin. Then, evidently, the family of clines orthogonal to that is the family of concentric circles whose center is the origin. Then apply the inverse transformation to get a family of circles orthogonal to the original family. The result is the family of Steiner circles of the second kine orthogonal to the given family of the first kind.

The normal form of a Möbius transformation. We saw before that, except for the identity transformation, every Möbius transformation has either 1 or 2 fixed points. We'll study those with two fixed points now.

Let T be a Möbius transformation with two fixed points, p and q. We can conjugate T by the transformation S mentioned above, $S(z) = \dfrac{z-p}{z-q}$ which maps $p \mapsto 0$ and $q \mapsto \infty$. That will give us a transformation $R = STS^{-1}$. Since T has fixed points p and q, therefore R will have fixed points 0 and ∞.

But it's easy to see what Möbius transformations R fix 0 and ∞. They're all of the form $R(z) = \lambda z$, where λ is a nonzero complex constant. We can interpret R as the composition of a rotation around the origin and a scaling by a nonzero real factor fixing the origin.

Now, a little algebra allows us to put T into a "normal form." From the equation $ST = RS$, we can write

$$\frac{T(z)-p}{T(z)-q} = \lambda \frac{z-p}{z-q}.$$

This is a particularly nice way to describe T since it involves the two fixed points p and q of T along with the complex constant λ that includes the rotation angle and the scaling factor.

Elliptic, hyperbolic, and loxadromic transformations. We can classify the Möbius transformations with two fixed points into these three kinds.

When $|\lambda| = 1$, the transformation T is called an *elliptic transformation*. Here, λ describes a rotation by an angle θ, that is, $\lambda = e^{i\theta}$. The conjugate R of T is the rotation of the plane about the origin by the angle θ, but T describes some kind of swirling about the fixed points p and q that moves points along the orthogonal family of Steiner circles of the second kind. Using the stereographic projection to the sphere, we can see that T is something like a rotation the sphere by an angle of θ fixing the points corresponding to p and q in $\mathbf{C}^+$, but it's not an actual rotation of the sphere unless those two points are antipodal points on the sphere.

When λ is a positive real number, the transformation T is called a *hyperbolic transformation*. The conjugate R of T is a scaling of the plane fixing the origin. The transformation T moves points along the familty of Steiner circles of the first kind that pass through the fixed points p and q.

When λ is not of the previous two special cases, it is the product of two complex numbers, one being $\frac{\lambda}{|\lambda|}$ which is a unit complex that describes a rotation, the other being the real positive number $|\lambda|$ which describes a scaling. The transformation T is called *loxadromic*, and it's a composition of a elliptic transformation and a hyperbolic transformation with the same two fixed points.

Transformations with one fixed point. Now that we've examined all the transformations with two fixed points, let's look at the ones with only one fixed point. They're called parabolic transformations.

Let T be a transformation with only one fixed point p. We can map p to ∞ with the transformation $S(z) = \dfrac{1}{z-p}$. Then the circles that pass through p will be mapped to straight lines (which all pass through ∞).

We can conjugate T by the transformation S. That will give us a transformation $R = STS^{-1}$ which has the formula $R(z) = z + \beta$, where β is a complex constant. Since T has the fixed point p, therefore R will have fixed point ∞. Indeed, R is a translation by the complex number β. It translates along a family of lines parallel to the direction of β.

If we apply the inverse transformation S^{-1} to this family of parallel lines, we'll get a family of circles that pass through the point p and are all tangent to each other. This family of circles is sometimes called a *degenerate family of Steiner circles*. The family of curves orthogonal to the degenerate family of Steiner circles is another one.