Math 128, Modern Geometry

D. Joyce, Clark University

19 Sep 2005

Last time. Began Möbius geometry.

Due Friday. Exercises from chapter 5: 1, 2, 3a, 4, 6 (first two), 8a.

First test. Friday, September 30.

Today. More on Möbius geometry, especially cross ratios.

Fixed points of a Möbius transformation. It will help us to understand Möbius transformations if we have some understanding of their fixed points. A *fixed point* z of a transformation T is a point such that $T(z) = z$.

Of course, every point is a fixed point of the identity transformation $I(z) = z$, but we'll see in a moment that no other Möbius transformation has more than two fixed points. Suppose

$$T(z) = \frac{az+b}{cz+d} = z.$$

A little algebra simplifies this equation to

$$cz^2 + (d-a)z - b = 0.$$

This is a nontrivial equation when $T \neq I$, and it's either a quadratic or linear equation depending on whether c is nonzero or zero. Therefore, it has at most two solutions (since a quadratic equation either has two solutions or one double solution, while a linear equation has one solution). That means, excepting the identity transformation, a Möbius transformation has at most two fixed points.

Cross ratios. The *cross ratio* of four complex numbers z_0, z_1, z_2, and z_3 is defined to be

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3}\,\frac{z_1 - z_3}{z_1 - z_2}.$$

You may find the definition of cross ratio easier to remember if you write it in the form

$$(z_0, z_1, z_2, z_3) = \frac{z_0 - z_2}{z_0 - z_3}\Big/\frac{z_1 - z_2}{z_1 - z_3}.$$

Actually, we rarely need this definition once we have the observation in the next paragraph.

When z_1, z_2, and z_3 are distinct, and z_0 is replaced by a variable z, this cross ratio can be used to define a Möbius transformation

$$T(z) = (z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3}\,\frac{z_1 - z_3}{z_1 - z_2}.$$

Note that $T(z_1) = 1$, $T(z_2) = 0$, and $T(z_3) = \infty$. We'll also express $T(z_1) = 1$ by saying that T maps z_1 to 1. Symbolically T maps

$$z_1 \mapsto 1, z_2 \mapsto 0, z_3 \mapsto \infty.$$

The following theorem is sometimes called *The Fundamental Theorem of Möbius Geometry.*

Theorem: If z_1, z_2, and z_3 are three distinct complex numbers, and w_1, w_2, and w_3 are also three distinct complex numbers, then there is a unique Möbius transformation that maps each z_i to w_i:

$$z_1 \mapsto w_1, z_2 \mapsto w_2, z_3 \mapsto w_3.$$

Proof: First we'll show that there is at least one by exhibiting it. Simply take the composition of the Möbius transformation (z, z_1, z_2, z_3) and the inverse of (z, w_1, w_2, w_3). Since the first one maps

$$z_1 \mapsto 1, z_2 \mapsto 0, z_3 \mapsto \infty,$$

while the inverse of the second one maps

$$1 \mapsto w_1, 0 \mapsto w_2, \infty \mapsto w_3,$$

therefore their composition accomplishes the goal.

Second, we'll show that there's only one such transformation. Suppose, on the contrary, that U and T each map

$$z_1 \mapsto 1, z_2 \mapsto 0, z_3 \mapsto \infty.$$

Then the composition of T with the inverse of U, that is, the transformation $U^{-1}T$, maps

$$z_1 \mapsto z_1, z_2 \mapsto z_2, z_3 \mapsto z_3.$$

That means $U^{-1}T$ has at least 3 fixed points. But only the identity transformtion I has more than 2 fixed points. Therefore, $U^{-1}T = I$. But that implies $T = U$. Therefore, there's only one such transformation. Q.E.D.

One of the implications of this theorem is that every Möbius transformation is of the form

$$T(z) = (z, z_1, z_2, z_3) = \frac{z - z_2}{z - z_3} \frac{z_1 - z_3}{z_1 - z_2}.$$

That's because this theorem says the Möbius transformation is determined by what three complex numbers z_1, z_2, and z_3 are sent to 1, 0, and ∞, respectively.

Invariance of the cross ratio. Recall that a function is an invariant of a geometry if it has the same value for congruent figures, that is, if A is a figure and T a transformation of the geometry, then the function has the same values for A and $T(A)$.

We can treat the cross ratio (z_0, z_1, z_2, z_3) as a function of a 4-point figure. (Strictly speaking, the four points are not a subset of $\mathbf{C}^+$, but an ordered subset of $\mathbf{C}^+$, or perhaps you could label the subset with labels 1, 2, 3, and 4. We really do want to be able to have labels on our figures, so we should probably go back and modify our defintion of figure.) To show that it's an invariant of Möbius geometry, we have to show that the cross ratio of this 4-point figure has the same value as the cross ratio of the image of this 4-point figure under any Möbius transformation T, that is, we have to show

$$(z_0, z_1, z_2, z_3) = (T(z_0), T(z_1), T(z_2), T(z_3)).$$

Theorem. The cross ratio is invariant in Möbius geometry.

Proof: Let S be the transformation that maps

$$z_1 \mapsto 1, z_2 \mapsto 0, z_3 \mapsto \infty,$$

that is, $S(z) = (z, z_1, z_2, z_3)$. Then the composition ST^{-1} maps

$$T(z_1) \mapsto 1, T(z_2) \mapsto 0, T(z_3) \mapsto \infty.$$

But the unique Möbius transformation that does that is $(z, T(z_1), T(z_2), T(z_3))$. Therefore, $ST^{-1}(z) = (z, T(z_1), T(z_2), T(z_3))$. Thus,

$$\begin{aligned} (z_0, z_1, z_2, z_3) &= S(z_0) \\ &= ST^{-1}(T(z_0)) \\ &= (T(z_0), T(z_1), T(z_2), T(z_3)) \end{aligned}$$

In words, the cross ratio is an invariant funtion of Möbius geometry. Q.E.D.

Now that we have this important invariant, let's see what it means in familiar geometric terms.

Clines. (Usually these are called circles or cycles, but we'll follow the text and call them clines.) Before giving the definition, let's have a theorem that justifies it.

Theorem. The cross ratio (z_0, z_1, z_2, z_3) is a real number if and only if the four points z_0, z_1, z_2, and z_3 either line on a Euclidean circle or on a straight line.

(Refer to the text for a proof.)

Definition. A *cline* in $\mathbf{C}^+$ (the complex plane with a point at ∞) is either a circle or a straight line, and when it's a straight line, it includes the point at ∞.

The preceeding theorem implies that set of clines is an invariant set of figures in Möbius geometry, that is, a Möbius transformation sends each cline to some other cline. Except for the identity transformation, every Möbius transformation will send some circles to straight lines and some straight lines to circles.