Math 128, first test sample problems.

These are sample questions. You may bring one sheet of prepared notes for the test. Calculators are optional; you may bring one if you like.

Problem 1. On complex numbers.

a. Write the reciprocal of $5 + 12i$ in the form $a + bi$.

b. What are the two square roots of i?

c. Write $e^{i\pi/3}$ in the form $a + bi$.

d. Write $5 + 5i$ in polar form $re^{i\theta}$.

Problem 2. On transformations.

a. Let S and T be two translations on the complex plane $\mathbf{C}$. Prove that their composition $T \circ S$ is also a translation.

b. The transformation $T(z) = iz + (1 - i)$ is a rotation. Determine (1) its fixed point, and (2) its angle of rotation.

c. Give a formula for the transformation T of the complex plane $\mathbf{C}$ which is a scaling (that is, a homothetic transformation) that fixes 0 and sends $3i$ to $4i$.

Problem 3. On the stereographic projection. Recall that with the stereographic projection the point (a, b, c) on the unit sphere $S^2 = \{(x, y, z) \,|\, x^2 + y^2 + z^2 = 1\}$ corresponds to the point $z = \dfrac{a + ib}{1 - c}$ in $\mathbf{C}^+$. The only exception to this correspondence is that the "North pole" $(0, 0, 1)$ of the sphere S^2 corresponds to the point ∞ in $\mathbf{C}^+$.

The plane $x + y + z = 1$ in space $\mathbf{R}^3$ intersects the unit sphere S^2 in a circle. By means of the stereographic projection, that circle corresponds to some curve in $\mathbf{C}^+$. Describe either in words or by means of an equation what curve that is. Explain your answer.

Problem 4. On transformation groups. Consider the collection G of transformations of the plane $\mathbf{C}$ that includes all translations and all half turns.

a. In order for G to be a transformation group, three conditions have to be satisfied. State each condition, and after you state it, explain why G satisfies that condition.

b. Let ABC be an equilateral triangle. Describe an equilateral triangle DEF whose sides are equal to those of ABC, but, with respect to the group G, triangle DEF is *not* congruent to triangle ABC.

Problem 5. On Möbius transformations.

a. There is one Möbius transformation T that maps $0 \mapsto 1$, $\infty \mapsto 0$, and $1 \mapsto \infty$. What is it?

b. Determine the fixed points of the transformation T you found in part a.

Problem 6. Prove that the only Möbius transformations that have a single fixed point at ∞ are translations.