Assignment 2: Concurrence, a Measure of Entanglement
DUE: Thursday, 2/23/2023, work in pairs

A little background: Let's consider an arbitrary 2-qubit state

$$|\Psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle.$$

On page 18, Mermin remarks in passing that the state $|\Psi\rangle$ is separable iff $\alpha\delta - \beta\gamma = 0$ (actually, at that point he's using the notation α_{00} for α, α_{01} for β, etc., but for our purposes, I expect you will welcome the typographical convenience of dispensing with subscripts). This is not as obvious as the casual remark may indicate! It's also quite an interesting fact. The quantity $2|\alpha\delta - \beta\gamma|$ is called the *concurrence*. We will derive some of its important properties, and then apply it to some cases. This leads to the important fact that some states are more entangled than others; it's not quite a dichotomy!

The first two problems are a little challenging, so I'll provide you with what I found to be some crucial insights (full disclosure: I struggled to find these myself!). Of course, if you find a better way to prove these facts, I'd like to hear it.

1. Show that the concurrence ranges between 0 and 1. That is, prove that

   $$0 \le 2|\alpha\delta - \beta\gamma| \le 1,$$

   i.e., $0 \le |\alpha\delta - \beta\gamma| \le 1/2$.

   HINTS: Consider the related state $|\Phi\rangle = \delta^*|00\rangle - \gamma^*|01\rangle - \beta^*|10\rangle + \alpha^*|11\rangle$. Ask yourself: (a) What upper and lower bounds can you put on $|\langle\Phi|\Psi\rangle|^2$ (and indeed for any two single- or multiple- qubit states $|\Phi\rangle$ and $|\Psi\rangle$, quite independent of this problem!)? (b) What do I get if I compute $\langle\Phi|\Psi\rangle$ in terms of $\alpha, \beta, \gamma, \delta$? (c) From the answers I get to (a) and (b), what can I conclude about the upper and lower bounds on $|\alpha\delta - \beta\gamma|$?

In the next two problems, we will prove that $|\Psi\rangle$ is separable *if and only if* $\alpha\delta - \beta\gamma = 0$. That is, it is possible to write $|\Psi\rangle = (a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$, where $|a|^2 + |b|^2 = |c|^2 + |d|^2 = 1$, iff $\alpha\delta - \beta\gamma = 0$. For the "iff" we of course need two logical directions, the "if" and the "only if."

2. The "only if" direction: Prove that if $|\Psi\rangle$ is separable, then $\alpha\delta - \beta\gamma = 0$. This is relatively easy, with the below hints.
   HINTS: If $|\Psi\rangle$ is separable, then we can write it as a tensor product, e.g., $(a|0\rangle + b|1\rangle) \otimes (c|0\rangle + d|1\rangle)$. Expand this, work out that $\alpha, \beta, \gamma, \delta$ are in terms of a, b, c, d, and show that the concurrence is then 0.

3. Now prove the "if" direction: that is, if $\alpha\delta - \beta\gamma = 0$, then $|\Psi\rangle$ is separable. This is not so easy, so more extensive hints here.
   HINTS: First prove that if any one of $\alpha, \beta, \gamma, \delta$ is 0, then (assuming, as we do, that $\alpha\delta - \beta\gamma = 0$) the state is separable. There are 4 cases here; it is enough to prove one, and if your argument is sufficiently air-tight, I will accept the assertion that "the other cases

are similar." Now turn to the remaining possibility that $\alpha, \beta, \gamma, \delta$ are all non-zero. I hope it's obvious to you that, in such a case, we can write $|\Psi\rangle = \frac{\alpha}{\gamma}\gamma|00\rangle + \frac{\beta}{\delta}\delta|01\rangle + \gamma|10\rangle + \delta|11\rangle$. This is a more significant observation than you may imagine! Starting with this simple observation, *use the hypothesis* (i.e., $\alpha\delta = \beta\gamma$) to change this into a form that can be written as a (tensor) product of two states. Now that might sound like you're done, but the problem is that if you do this naively, you will find that the two "states" aren't normalized to 1, so they're not really qubits. At that point, you must show how to manipulate quantities so that you get a tensor product of two genuine normalized qubit states. (Caution: proving the two qubits in the product are normalized will likely require another use of the hypothesis.) At that point, you're done.

4. Let's *do* something with the concurrence! Since $0 \leq 2|\alpha\delta - \beta\gamma| \leq 1$, where the "0" value corresponds to "separable" (as we saw in the last result), it makes sense to regard concurrence as a *measure of entanglement*. We say a state with concurrence 1 is *maximally entangled*. Let's say if it's somewhere strictly between 0 and 1 it's *partially entangled*; and of course if it is 0, the state is not at all entangled, i.e., separable. For each of the following 2-qubit states, identify α, β, γ, and δ, compute the concurrence, and state whether the state is separable, partially entangled, or maximally entangled. *For the separable states, express the state as a tensor product of 2 single-qubit states.* Of those states here that are entangled, which one(s) is (are) the *least* entangled?

   (a) $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)$
   (b) $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - |11\rangle\right)$
   (c) $\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle - i|11\rangle\right)$
   (d) $\frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |11\rangle\right)$
   (e) $\frac{1}{\sqrt{2}}\left(|00\rangle - |10\rangle\right)$
   (f) $\frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)$