Assignment 1

DUE: Tuesday, 2/7/2023, work individually

1. Derive Eq. (1.38) in two ways: (a) Using matrix multiplication from the definitions of X and Z, and (b) Algebraically, that is, not using matrix multiplication, only the relations $\mathsf{Z} = \tilde{\mathsf{n}} - \mathsf{n}$, $\mathsf{X}\mathsf{n} = \tilde{\mathsf{n}}\mathsf{X}$, and $\mathsf{X}\tilde{\mathsf{n}} = \mathsf{n}\mathsf{X}$.

2. Beginning with the form for S_{ij} in Eq. (1.35), prove, that $\mathsf{S}_{ij}^2 = 1$. Do *not* use the matrix form for S_{ij}. Rather, derive this algebraically using only the properties of $\tilde{\mathsf{n}}$, n, and X as in Eqs. (1.33) and (1.34), and the fact that $\mathsf{X}^2 = 1$. Remember that as the n's and X are all single-Cbit operators, and by virtue of the positional notation (the i's and j's), those with different indices *commute* in tensor products: That is, $\mathsf{X}_i\mathsf{X}_j = \mathsf{X}_i \otimes \mathsf{X}_j = \mathsf{X}_j \otimes \mathsf{X}_i = \mathsf{X}_j\mathsf{X}_i$, and similarly, $\mathsf{X}_i\mathsf{n}_j = \mathsf{n}_j\mathsf{X}_i$, etc. On the other hand, those with the same index do *not* necessarily commute, e.g., $\mathsf{n}_i\mathsf{X}_i = \mathsf{X}_i\tilde{\mathsf{n}}_i$, as dictated by Eq. (1.34).

3. Along similar lines to problem (1), prove the identity $\mathsf{S}_{ij} = \mathsf{C}_{ij}\mathsf{C}_{ji}\mathsf{C}_{ij}$ (Eq. (1.23)) in two ways:

   (a) Straightforwardly, using the matrix representations of S_{ij} and C_{ij}.

   (b) Using only using the identity given in Eq. (1.36), i.e., $\mathsf{C}_{ij} = \tilde{\mathsf{n}}_i + \mathsf{X}_j\mathsf{n}_i$ and the properties of the n's and X as in the previous problem. Follow the hint given by Mermin on page 13: Substitute the formula for C_{ij} given in Eq. (1.36) in Eq. (1.23). You should find that of the resulting 8 terms, 4 of the terms vanish, and the remaining form yield the formula for S_{ij} given in Eq. (1.35).

   [Full disclosure: In this case, you will certainly find the technique of part (a) to be easier! But the technique of part (b) will help you understand the notation far more deeply, and provides valuable algebraic practice with tools that are more easily generalized to larger matrices. Similar comment for problem (2): deriving the formula directly from matrix multiplication of the 4×4 matrices is far easier.]

4. Write down the explicit 4×4 matrix for the tensor product of the Hadamard operator with itself, i.e., $\mathsf{H} \otimes \mathsf{H}$. Tensor products of operators *as matrices* is discussed in lecture, although it is not spelled out in the book. To build a tensor product of two 2×2 matrices, in general, you multiply each element of the first matrix times the *entire* second matrix, which gives four 2×2 blocks which, when stacked together as bricks, form a 4×4 matrix. Here's what I mean (for *any* a, b, c, d, e, f, g, h):

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \otimes \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} a\begin{bmatrix} e & f \\ g & h \end{bmatrix} & b\begin{bmatrix} e & f \\ g & h \end{bmatrix} \\ c\begin{bmatrix} e & f \\ g & h \end{bmatrix} & d\begin{bmatrix} e & f \\ g & h \end{bmatrix} \end{bmatrix} = \begin{bmatrix} ae & af & be & bf \\ ag & ah & bg & bh \\ ce & cf & de & df \\ cg & ch & dg & dh \end{bmatrix}$$

Simply apply this formula to two H's; you should get 4 rows, all consisting only of 1's and -1's. Also note there is a normalizing factor of $\frac{1}{\sqrt{2}}$ in H. Be sure to work out the corresponding normalizing factor in $\mathsf{H} \otimes \mathsf{H}$.

Once you have an exact form for $\mathsf{H} \otimes \mathsf{H}$, show that each row and column is orthogonal. (This is a nice sanity check that your answer is correct. In fact, observe that is the case for H itself.)

5. Using matrices in a straightforward manner, and the result of the previous problem, verify the relation $\mathsf{H}_i\mathsf{H}_j\mathsf{C}_{ij}\mathsf{H}_i\mathsf{H}_j = \mathsf{C}_{ji}$. Note that here $\mathsf{H}_i\mathsf{H}_j$ does not mean the product of two matrices, but is rather a shorthand for the tensor product $\mathsf{H}_i \otimes \mathsf{H}_j$, as worked out in problem 4. So I am asking you to compute the product of three 4×4 matrices.