The Book Review Column[1]
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There are plenty of books available for reviewing, and the list is not exhaustive. If you're interested, please send me an email, drop me a line, stating point of view! In this column, there are 4 books on the menu: We start with some applied algebraic topology, go for a quantum walk, and, in a joint review, traverse most of basic math, ending with a bit of category theory for dessert.

1. **The Structure and Stability of Persistence Modules,** by Frédéric Chazal, Vin de Silva, Marc Glisse, Steve Oudot. A book about a hot topic in applied algebraic topology. Review by Robin Belton and Brittany Terese Fasy.

2. **Quantum Walks and Search Algorithms,** by Renato Portugal. A detailed introduction to a fascinating phenomenon, which has many applications in quantum computing. Review by Frederic Green.

3. **The Magic of Math,** by Arthur Benjamin (a magical introduction to everything), and **How to Bake** π**,** by Eugenia Cheng (an edible introduction to category theory). Joint review by Frederic Green

BOOKS THAT NEED REVIEWERS FOR THE SIGACT NEWS COLUMN

Algorithms

1. *Distributed Systems: An algorithmic approach (second edition)*, by Ghosh
2. *Tractability: Practical approach to Hard Problems*, Edited by Bordeaux, Hamadi, Kohli
3. *Recent progress in the Boolean Domain*, Edited by Bernd Steinbach
4. *A Guide to Graph Colouring Algorithms and Applications*, by R.M.R. Lewis
5. *Twenty Lectures on Algorithmic Game Theory,* by Tim Roughgarden
6. *Compact Data Structures,* by Gonzalo Navarro
7. *Algorithms and Models for Network Data and Link Analysis,* by François Fouss, Marco Saerens, and Masashi Shimbo

Programming Languages

1. *Selected Papers on Computer Languages* by Donald Knuth
2. *Practical Foundations for Programming Languages,* by Robert Harper

Miscellaneous Computer Science

1. *Algebraic Geometry Modeling in Information Theory,* Edited by Edgar Moro
2. *Communication Networks: An Optimization, Control, and Stochastic Networks Perspective* by Srikant and Ying
3. *CoCo: The colorful history of Tandy's Underdog Computer* by Boisy Pitre and Bill Loguidice
4. *Introduction to Reversible Computing*, by Kalyan S. Perumalla
5. *A Short Course in Computational Geometry and Topology*, by Herbert Edelsbrunner
6. *Network Science*, by Albert-László Barabási
7. *Actual Causality*, by Joseph Y. Halpern
8. *Partially Observed Markov Decision Processes,* by Vikram Krishnamurthy
9. *The Power of Networks*, by Christopher G. Brinton and Mung Chiang
10. *Game Theory, Alive,* by A. Karlin and Y. Peres

Computability, Complexity, Logic

1. *The Foundations of Computability Theory,* by Borut Robič
2. *Models of Computation*, by Roberto Bruni and Ugo Montanari
3. *Proof Analysis: A Contribution to Hilbert's Last Problem* by Negri and Von Plato.

Cryptography and Security

1. *Cryptography in Constant Parallel Time,* by Benny Appelbaum
2. *Secure Multiparty Computation and Secret Sharing*, Ronald Cramer, Ivan Bjerre Damgård, and Jesper Buus Nielsen

3. *A Cryptography Primer: Secrets and Promises*, by Philip N. Klein

Combinatorics and Graph Theory

1. *Finite Geometry and Combinatorial Applications*, by Simeon Ball
2. *Introduction to Random Graphs*, by Alan Frieze and Michał Karoński
3. *Erdös–Ko–Rado Theorems: Algebraic Approaches*, by Christopher Godsil and Karen Meagher
4. *Combinatorics, Words and Symbolic Dynamics,* Edited by Valérie Berthé and Michel Rigo
5. *Words and Graphs*, by Sergey Kitaev and Vadim Lozin

Miscellaneous Mathematics and History

1. *Professor Stewart's Casebook of Mathematical Mysteries* by Ian Stewart

Review[2] of
The Structure and Stability of Persistence Modules
by Frédéric Chazal, Vin de Silva, Marc Glisse, Steve Oudot
Springer, 2015
113 **pages, Softcover, US$54.99**

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1 Introduction

The Structure and Stability of Persistence Modules provides a comprehensive introduction to the theory of persistence in applied algebraic topology. The book is divided into six parts: Introduction, Persistence Modules, Rectangle Measures, Interleaving, The Isometry Theorem, and Variations. After providing a brief history and overview of the field, the authors dive into the definition of a persistence module and discuss its relations to persistence diagrams and notions of distances between persistence modules throughout the book.

The target audience for this book is both newcomers and experts alike, providing both a foundation for newcomers to build off of and a general framework for experts. In particular, we believe this book is most accessible to graduate students in mathematics, who already have some familiarity with the idea of persistence.

Overall, the authors capture the attention of the audience through the abundance of motivating questions, detailed figures, and hints towards applications of the theory, while maintaining the necessary rigor within definitions and proofs.

2 Summary

The Structure and Stability of Persistence Modules by Chazal, de Silva, Glisse, and Oudot is a concise introduction to the field of applied algebraic topology. Specifically, this book explores persistent homology using continuous parameters. We briefly summarize the chapters of the book.

Chapter 1: Introduction

The first chapter sets the stage for the general context of this book, giving an overview of what the book will cover and starting with a brief history and literature review of the subject area, from the independent work of Frosini [5] and Robins [7] to the birth of the term *persistent homology* [4] to recent results at the time of this book's publication (e.g., [1, 2, 6]).

Chazal et al. are interested in the mathematical properties of persistence modules indexed by a single continuous parameter. In particular, they aim show that a continuous parameter can be handled just as effectively as a discrete parameter (the latter being the setting under which persistence theory is traditionally studied).

In this chapter, two applications (stable descriptors and stable clustering) are used in order to motivate the study of *the structure and stability of persistence modules.* These applications illustrate the usefulness of the persistence diagram for problems in data analysis, machine learning, and clustering models.

Chapter 2: Persistence Modules

The authors define persistence modules in a few different contexts. They first define persistence modules in terms of vector spaces, then over posets, and lastly describe how they can be viewed as categories.

A discrete persistence module is a sequence of vector spaces connected by linear maps:

$$V_1 \to V_2 \to \ldots \to V_n,$$

such that the composition of two linear maps is defined. This book considers the more general setting; a continuous persistence module is a collection of vector spaced indexed by $\mathbb{R}$: $(V_t)_{t\in\mathbb{R}}$, where V_t is a vector space for all t. Let v_j^i denote the unique linear map $V_i \to V_j$ for $i \leq j \in \mathbb{R}$, and r_j^i denote the rank of v_j^i. In particular, one commonly studied setting is where V_i are sub-level sets of a function $f\colon \mathbb{X} \to \mathbb{R}$ for some topological space $\mathbb{X}$, and the vector spaces are homology groups: $V_i = H_k(f^{-1}((-\infty, r_i])$. All of the information contained in $\{v_{i,j}\}$ can be encoded in what is called the *persistence diagram or barcode*, which is a decomposition of the persistence module into intervals representing different homology generators.

While discussing persistence modules within these contexts, the authors discuss the persistence diagram or barcode that is commonly used in applied algebraic topology/topological data analysis. They then discuss how and under what circumstances one can understand a persistence module by decomposing it into intervals (bringing up the notation of decorated reals and interval decomposition). They introduce the notion of a decorated diagram, which can be used to differentiate open intervals (a, b) from closed intervals $[a, b]$ and half-open intervals $(a, b]$ and $[a, b)$. Lastly, Chazal et al. introduce quiver notation, which is a representation of a diagram of n vector spaces and $n-1$ linear maps that represent a persistence module $\mathbb{V}$. Note that, for this review, we avoid introducing the notation for decorated persistence diagrams and quivers.

Chapter 3: Rectangle Measures

In the longest chapter of the book (31 pages), the authors establish an equivalence between persistence diagrams and an integer-valued measure on rectangles. This addresses the question of how to define the diagram of a persistence module in the case where the module does not decompose into intervals. The measure is defined as follows: Let $\mathbb{V}$ be a persistence module, and let Rect($\mathbb{R}$) denote the set of all rectangles $[a, b] \times [b, c]$ with $a < b \leq c < d \in \mathbb{R}$. The **persistence measure** of $\mathbb{V}$ is the function $\mu_{\mathbb{V}}\colon \text{Rect}(\mathbb{R}) \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ defined by

$$\mu_{\mathbb{V}}([a, b] \times [c, d]) = \langle \circ_a - \bullet_b - \bullet_c - \circ_d \mid \mathbb{V}\rangle = r_c^b - r_c^a - r_d^b + r_d^a,$$

where $\langle \circ_a - \bullet_b - \bullet_c - \circ_d \mid \mathbb{V}\rangle$ is the multiplicity of $\mathbf{k}[b, c]$ in the four-term module: $\mathbb{V}_{a,b,c,d} : V_a \to V_b \to V_c \to V_d$. This measure is additive with respect to splitting a rectangle into two rectangles rather than the usual sense of disjoint unions.

The authors then continue to describe rectangle measures more abstractly. They define a rectangle measure as follows:

Let $\mathcal{D}$ be a subset of $\mathbb{R}^2$. Define

$$\text{Rect}(\mathcal{D}) = \{[a,b] \times [c,d] \subset \mathcal{D} \mid a < b \text{ and } c < d\}.$$

(the set of closed rectangles contained in $\mathcal{D}$). A **rectangle measure** or **r-measure** on $\mathcal{D}$ is a function

$$\mu : \text{Rect}(\mathcal{D}) \to \{0, 1, 2, ...\} \cup \{\infty\}$$

that is additive under vertical and horizontal splitting.

Theorem 3.12 (The equivalence theorem) states that for $\mathcal{D} \subseteq \mathbb{R}^2$, there is a bijective correspondence between finite r-measures on $\mathcal{D}$ and locally-finite multi-sets in $\cup_{R \in \text{Rect}(\mathcal{D})} R$. For every $R \in \text{Rect}(\mathcal{D})$, the measure μ corresponding to a multiset A is related to it by the formula $\mu(R) =$card$(A|_R)$.

The authors also go into detail on how to deal with r-measures that are not finite everywhere and in the extended plane. In particular, the measure persistence diagrams of a persistence module $\mathbb{V}$ are obtained by defining its persistence measure $\mu_{\mathbb{V}}$ on the extended half-plane $\overline{\mathcal{H}}$.

Using these notions, the authors establish that given a persistence module, the persistence diagram exists whenever the measure takes finite values. This leads to the notion of "tameness" in §3.8. There are many natural occurring examples of persistence modules which are tame enough for their diagrams to be defined everywhere or almost everywhere. The authors are mostly concerned with four types of tameness: q-tame, h-tame, v-tame, and r-tame, which discern the finiteness of $\mu_{\mathbb{V}}$ over different types of rectangles not touching the diagonal $y = x$: quadrants, horizontal strips, vertical strips, and bounded rectangles, respectively. More specifically, we say that $\mathbb{V}$ is ***q*-tame**, if $\mu_{\mathbb{V}}(Q)$ is finite for every quadrant Q not touching the diagonal.

One can show the following implications with these definitions:

$$q\text{-tame} \Rightarrow (h\text{-tame and } v\text{-tame}), \quad (h\text{-tame or } v \text{ -tame}) \Rightarrow r\text{-tame}.$$

The proof of these two implications is a straightforward exercise.

In §$3.10 - 11$, the authors discuss lemmas that guarantee the vanishing of the persistence diagram in certain parts of the plane that help simplify the task of computing a persistence diagram given a persistence module. The chapter ends by relating the measure-theoretic persistence diagrams to the diagrams constructed more traditionally, mostly being by a computer with finite information.

Chapter 4: Interleaving

Chapter 4 investigates a notion of distance between persistence modules: the interleaving distance. One motivating reason for studying distances between persistence diagrams is the need to describe two persistence modules as being *close* so that we have a way of stating that a diagram computed from data - under the right conditions - results in a diagram *close* to the 'ground truth' diagram, since recovering the actual ground truth would be quite unlikely.

Let $u^i_j \colon U_i \to U_j$ and $v^i_j \colon V_i \to V_j$ be the linear maps between vector spaces of persistence modules $\mathbb{U}$ and $\mathbb{V}$, respectively. The modules $\mathbb{U}$ and $\mathbb{V}$ are said to be δ-*interleaved* if for all $t \in \mathbb{R}$, there exists maps $\phi_t \colon U_t \to V_{t+\delta}$ and $\psi_t \colon V_t \to U_{t+\delta}$ for all t such that these maps commute with the linear operators of the modules:

$$\phi_j \circ u^i_j = v^{i+\delta}_{j+\delta} \circ \phi_i, \text{ and } \psi_j \circ v^i_j = u^{i+\delta}_{j+\delta} \circ \psi_i, \tag{1}$$

and

$$u^{i+\delta}_{i-\delta} = \psi_i \circ \phi_{i-\delta}, \text{ and } v^{i+\delta}_{i-\delta} = \phi_i \circ \psi_{i-\delta}, \tag{2}$$

for all $i \leq j$ and $\delta > 0$. These equalities are summarized as commutative diagrams in Section 4.2. The chapter goes on to prove (in Theorem 4.7) that if the modules are δ-interleaved, then there exists a path of length δ in the space of persistence moduels connecting $\mathbb{U}$ and $\mathbb{V}$.

Chapter 5: The Isometry Theorem

In Chapter 5, the authors prove the Isometry Theorem - that the interleaving and bottleneck distances are equal when the persistence modules are q-tame. They begin by defining the interleaving distance.

The **interleaving distance** between two persistence modules is defined to be

$$\begin{aligned} d_i(\mathbb{U}, \mathbb{V}) &= \inf\{\delta \mid \mathbb{U}, \mathbb{V} \text{ are } \delta\text{-interleaved}\} \\ &= \min\{\delta \mid \mathbb{U}, \mathbb{V} \text{ are } \delta^+\text{-interleaved}\}. \end{aligned}$$

If there is no δ-interleaving between $\mathbb{U}, \mathbb{V}$ for any value of δ, then $d_i(\mathbb{U}, \mathbb{V}) := \infty$. Next, the Bottleneck Distance is introduced in $\S 5.2 - 3$. Before defining this distance, the notion of a partial matching and δ-matching are defined. A **partial matching** between multisets A and B is a collection of pairs

$$M \subset A \times B$$

such that no element in A or B appears in more than one pair. We say that a partial matching M is a δ-**matching** if $d^\infty(\alpha, \beta) \leq \delta$ for all $(\alpha, \beta) \in M$ and all unmatched points in A and B are within δ of the diagonal Δ.

Using these notions the **bottleneck distance** between two multisets A, B in the extended half-plane is defined to be

$$d_b(A, B) = \inf\{\delta \mid \text{there exists a } \delta\text{-matching between } A \text{ and } B\}.$$

With this definition in hand, the **Isometry Theorem** is stated as the following equality:

$$d_i(\mathbb{U}, \mathbb{V}) = d_b(\text{dgm}(\mathbb{U}), \text{dgm}(\mathbb{V})),$$

where $\mathbb{U}$, $\mathbb{V}$ are q-tame persistence modules. This equality is proven in two steps: the so-called *stability theorem*

$$d_i(\mathbb{U}, \mathbb{V}) \geq d_b(\text{dgm}(\mathbb{U}), \text{dgm}(\mathbb{V})),$$

and the *converse stability theorem*

$$d_i(\mathbb{U}, \mathbb{V}) \leq d_b(\text{dgm}(\mathbb{U}), \text{dgm}(\mathbb{V})).$$

The proof of these statements is covered in $\S 5.5 - 6$ and generalized in $\S 5.7 - 8$ for diagrams of abstract r-measures. The key idea in proving the converse is that persistence modules can be approximated by better-behaved persistence modules, using a procedure called *smoothing*. In order to prove the stability theorem, the authors use a constructive argument.

Chapter 6: Variations

The book concludes with a chapter discussing how to use persistence modules with sampled data, where the theoretical results presented thus far are too strong to employ. Chapter 6 is devoted to the proof of Theorem 6.1, which can be thought of a stability result for capturing the persistent homology of a density function above some minimum density. In particular, consider two q-tame persistence modules $\mathbb{U}$ and $\mathbb{V}$ such that there exists $t_0 > 0$ where (1) and (2) hold for all $i, j, t \leq t_0$. Then, there exists a bijection between the diagrams that matches all points sufficiently far enough from the diagram with birth times at most $t_0 - \delta$.

3 Opinion

The Structure and Stability of Persistence Modules provides a thorough introduction to the algebraic foundations of persistence. Extending upon a technical report with the same title [3], this book includes history, definitions, proofs, examples, and provides a good starting point for people who are entering the field.

We did not find many weaknesses within the book, as it is well-written, concise, and complete in the coverage of continuous-parameter persistence modules. However, we list three weaknesses here. First, this book is difficult to read for a subset of the intended audience: junior graduate students unfamiliar with the ideas of persistence modules, but interested in knowing more. In particular, the sections on distances between persistence modules provide very few examples demonstrating the distances, creating a barrier for a complete novice to the field to fully synthesize the information. In addition, the book does not contain any exercises that could help a junior graduate student gain the intuition necessary to pursue research in these topics. Second, since the field of computational topology - and specifically the topic of persistent homology - is a hot topic right now, new results are published regularly. So, this book could easily become dated. Lastly, the book is focused on the theory of persistence modules, and does not fully investigate the power of persistence modules in applications. Hence, if one is interested in the applications of persistence modules, this is not the right book to use.

Now, we highlight the strengths of the book. The authors are very clear in articulating what concepts they focus on and provide good motivation using applications and driving questions for studying the topics they present. The notation and proofs are rigorous, and the well-labeled figures and smooth transitions between chapters are very helpful in enhancing understanding of the abstract ideas. Moreover, this short book unites much of the current research in persistent homology seamlessly.

Overall, the book provides a nice story to the theory of persistence, while maintaining mathematical rigor. If you are interested in learning the theory of persistence and already have some understanding in the main ideas of the field, then we highly recommend this book to you.

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Review of[3]
Quantum Walks and Search Algorithms
by Renato Portugal
Springer, 2013
222 pages, Hardcover, US$69.95

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It is a well-known phenomenon that if you take a random walk along a line, it's more likely than not that you'll end up back where you started. The probability distribution over position after a sufficient amount of time follows a Gaussian centered on the starting point, and with high probability you won't get much more than $\sqrt{t}$ steps (the standard deviation) from that point in time t.

At any rate, that is what holds for random walks following the laws of *classical* probability theory. Quantum mechanics, of course, is weird. If you take a *quantum* walk, it's more likely than not that you *won't* end up where you started! And the standard deviation is now t, i.e., linear. The walker is spread more thinly and uniformly in the quantum case.

Now where have we seen this kind of "quadratic improvement" in going from classical to quantum before? In Grover's algorithm, of course, in which we find that a quantum search of an unsorted database of N elements can be done in $\sqrt{N}$ time, as opposed to the best classical performance of N. This is no coincidence: Grover's algorithm can be interpreted as a quantum walk on a complete graph. Roughly speaking, a quantum walker explores a search space much more widely, and in less time, than a classical walker.

Note that in the phrase "quantum walk," under current proper usage the word "quantum" really replaces the word "random" that is used in the classical case. One may be tempted to say "quantum *random* walk," but the unitary evolution that describes the process is perfectly deterministic. The randomness only comes in after one makes a measurement, so it is prudent not to fall into the trap of thinking that each step of a quantum walk is in any sense "random." It is "quantum."

These are the basic ideas introduced and extensively elaborated in Renato Portugal's detailed, focused and clearly-written monograph "Quantum Walks and Search Algorithms" ("QWSA"). One valuable aspect of QWSA is that the bulk of it studies an area of quantum computing ("QC") that is rarely covered in such depth anywhere else, and until now has been largely available only in the research literature. While Grover's algorithm (a key concept in the book) dates to 1996, most of the theory of quantum walks was developed since around 2001, so "classic" QC texts such as Nielsen & Chuang ([NC00]) or Mermin ([M07], reviewed in SIGACT News **41** (3), 2010) are either too early or too elementary to cover them at all, and the more recent texts that do get into quantum walks do so in less detail (Moore & Mertens [MM11], reviewed in SIGACT News **47** (1), 2016; Lipton & Regan [LR14], reviewed in SIGACT News **47** (3), 2016)).

After the introduction to QC in Chapter 2, the book gets more serious in Chapter 3, where it first reviews classical random walks. It then proceeds to discrete-time quantum walks, which provides an elementary context for introducing the primary ingredients in a quantum walk, i.e., the *coin operator* (which "randomly," or rather "quantumly," determines the direction of movement) and the *shift operator* (which implements the actual step). Thus the whole theory of quantum walks revolves around the unitary operator $U = S(C \otimes I)$, which represents one step of the quantum walk. Here S is the shift that operates on a register denoting the walker's spatial position, C is the "coin" operating on a register denoting the direction of the walker's step, and I the identity on the spatial register. Various models of walks are obtained by choosing the coin C (with some caveats, as is clarified in Chapter 5). Via symbolic or numerical solutions to the evolution equation for the state, one immediately sees the contrast with classical walks, characterized by the linear standard deviation in the distance from the origin (and also the low probability that the "walker" is found at the origin). A surprise along the way is that if one takes the coin operator to be the 2×2 Hadamard matrix H (the "Hadamard coin"), the distribution is not symmetric around the origin; this turns out to be an artifact of how the Hadamard interacts with the shift operator, which causes this choice of coin operator *not* to be symmetric. After a detour on classical continuous Markov chains, the chapter concludes with a similar calculation for a continuous quantum walk. In the remainder of the book, the emphasis is on discrete-time quantum walks.

Chapter 4 introduces Grover's search algorithm, first considering the case in which there is only one instance of the element being searched for. This begins as a relatively standard treatment, analyzing the algorithm in terms of reflection operators. However, this kind of analysis is not amenable to analytical calculations of quantum walks. For that purpose, we require the spectral decomposition of the unitary operators that drive the algorithm, and this is also presented. After proving the optimality of Grover's algorithm, the book proceeds to consider Grover's algorithm with repeated elements. Again, first this is analyzed using reflection operators, and then via spectral decomposition. Finally, the technique of amplitude amplification is presented, in which if an element can be found in one round with probability p, it can be found in $1/\sqrt{p}$ rounds with probability approaching 1, by contrast with the classical case in which the number of rounds is quadratically larger $(1/p)$.

Chapters 5 and 6 are mostly devoted to analytical solutions to the evolution equations for random walks on a variety of types of graphs. Chapter 5 considers infinite graphs, including the infinite line (recovering the numerical behavior found in Chapter 3) and the infinite square 2-dimensional lattice. Analytic solutions are not known for the latter, so numerical techniques are necessary. To that end, the book discusses the author's freely available simulation software *QWalk*. For the lattice, various choices of coin are considered, namely the Hadamard, the "Fourier," and the "Grover" coins. Interestingly, different choices of coin in the one-dimensional case can all be reproduced by adjusting the initial conditions; this is not true in the two-dimensional case. The Fourier Transform, and the spectral decomposition of U, are instrumental in obtaining analytic solutions in both the finite and infinite cases. This is a common and powerful technique in quantum computing, since the Fourier transform enables the spectral decomposition (i.e., diagonalizing U), which in turn makes it easy to raise U go any power, and thus compute the time evolution operator. Chapter 6 derives analytic solutions for finite graphs: the cycle, the finite two-dimensional lattice (with toroidal boundary conditions), and the hypercube. In the latter two cases, the shift operator flips the coin state, which leads to what is called a *flip-flop* quantum walk. For the hypercube, it turns out that the quantum walk can be interpreted as a quantum walk along a line, where the points along the line correspond to the Hamming weights on the hypercube. The pattern in all of these calculations is the same, although with diverse technical twists in each case: first take the Fourier Transform, then use the spectral decomposition of U to derive the time evolution operator exactly.

Classical random walks on connected non-bipartite graphs are known to have a limiting distribution which is, furthermore, independent of the initial conditions. Chapter 7 considers the analogous problem for quantum walks on finite regular graphs, and how long it takes to converge to such a distribution (the "mixing time"). As it happens, it's quite easy to see that a quantum walk does *not* have a limiting distribution (a direct consequence of unitarity is that the norm of the difference between the states at times t and $t+1$ is bounded away from zero). However, if we average the probability of (say) being at a particular vertex in the graph over time, we obtain a quantity, the *average probability distribution*, that *does* converge stochastically to a stationary distribution. This is the primary object of study in this chapter. The stationary distribution can be written in terms of the spectral decomposition of U, and can be seen to exist. After passing to the Fourier basis, drawing on results of the previous chapter, the author then specializes to obtain analytic expression for the limiting distributions for quantum walks in cycles, finite two-dimensional lattices, and hypercubes. These distributions result in mixing times for these structures that offer some dramatic speed-ups over the classical results, e.g., almost quadratic for cycles and lattices (ignoring other parameters like the error and logarithm in the number of vertices).

Chapters 3 through 7 are quite interesting in their own right, but the work done there has a payoff that comes in Chapter 8, on spatial search algorithms. Here we encounter the "abstract search algorithm," a generalization of Grover's algorithm for searching in a spatial structure such as a graph. A naive application of Grover's algorithm is not sufficient for obtaining a quadratic speed-up for finding a marked element on a two-dimensional lattice. But the idea of the abstract search algorithm is similar to the original Grover's algorithm, in that the vertex that is being searched for is "marked" by giving it a negative amplitude. However, as a model of a quantum walk, it differs in that the "coin" operator does not act the same on all vertices of the graph; a consequence of this is that the coin operator acts both on the coin space and on the vertex space. The resulting single-step quantum walk operator that generalizes the standard quantum walk operator U, here denoted U', is again to be applied an appropriate number of times to the initial state as in Grover's algorithm, and after that time we find the item we're searching for with high probability. Unfortunately, the spectral decomposition of U' (which again plays a key role) is no longer feasible even for simple graphs. On the other hand, the spectral decomposition of U (which is known from previous chapters for lattices and hypercubes) can be used to *approximate* the right power of U'. Via a careful analysis of this evolution operator, which uses results from Chapter 6, and amplitude amplification as described in Chapter 4, it is shown how to find an element in a finite 2d lattice of size N in time $O(\sqrt{N}\log N)$. The chapter concludes with the interpretation of Grover's algorithm as an abstract search algorithm on a complete graph (as mentioned near the outset of this review).

Finally, consider a connected, non-bipartite graph, and a subset M of the vertices. Classically, the *hitting time* for a random walk is the expected time for the walk to reach one of the vertices in M for the first time. A crucial element in Grover's algorithm is knowing when to stop applying the evolution operator, since if you keep going you "miss the mark"; the *quantum* hitting time basically tells you when to stop, since it determines when it is likely that a marked vertex will be found. Chapter 9 begins with a lucid review of the classical problem. In that setting the hitting time can be determined using the stationary distribution. Alternatively, it can be computed using a modified graph, in which marked vertices have loops that trap the walker once the vertex is reached, for which the stochastic matrices are different. The resulting method for determining the hitting time is more amendable to generalization to the quantum setting, which is considered next. In the quantum case a modified graph is again used, but now modified in two ways. In the first modification, a copy of the original set of vertices is made, partitioning the vertices into two disjoint sets. Edges only connect vertices in the original set with the copied set, so the graph is bipartite. In the second modification, as in the classical case, for a directed graph, edges leaving a marked vertex are converted to loops. This modification facilitates the construction of an evolution operator, derived from the stochastic matrix of the original graph. The evolution operator acts in terms of reflection operators on the two partitions of the graph, and in that sense is analogous to Grover's algorithm. At the same time, the bipartite construction eliminates the use of a coin operator as in the standard quantum walk. Techniques similar to those in earlier chapters enter into the determination of the quantum hitting time, e.g., spectral decomposition of the evolution operator, but also the singular value decomposition of a matrix (the discriminant) that relates the partitions of the graph. A rather elaborate analysis yields a hitting time which gives a quadratic speed-up compared to the classical case.

The material in QWSA is well-organized, and the exposition is systematic and pedagogically sound. It is virtually self-contained, including an introduction to quantum computing in the second chapter, and an appendix that reviews all the linear algebra that is needed for the main text. The text is interspersed with many exercises that engage the reader in the development, and it is eminently suitable for a focused course, a readings course, or for self-study, all at the advanced undergraduate or beginning graduate level. However, although it is an introduction to the subject, there are some who may find it tough going without prior exposure to quantum computing. The presentation is decidedly in the "physics" style: Rather than presenting results with formal definitions, theorems and proofs, the objects under study are introduced, the relevant expressions are presented or derived, calculations are done, and conclusions drawn. While this has the advantage of informality and narrative, it can take some work to determine what the discussion is driving at. Ultimately whether or not you like this style boils down to a matter of taste. Bottom line, I favor books that focus on specific problems within a unified framework. This is certainly such a book. It starts from the basics and gradually builds up to some quite interesting results. You will definitely want to read it if you have any interest in quantum walks.

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Joint Review of[4]
The Magic of Math
by Arthur Benjamin
Basic Books, 2015
321 pages, Hardcover, US$26.99

and

How to Bake π
by Eugenia Cheng
Basic Books, 2015
288 pages, Hardcover, US$27.50

Review by
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On occasion, I ponder what the various departments of Hogwarts' School of Witchcraft and Wizardry might correspond to in a real world school. Clearly, Potions would be Chemistry, Herbology would be Biology, and Astronomy would be... well, Astronomy. Maybe Charms would become Computer Science and Transfiguration would be Physics. But surely, Arithmancy would be a branch of Mathematics, probably Number Theory.

Yes, Mathematics can seem like a form of magic. The way things fall together in certain proofs is sometimes miraculous, and it's hard to avoid a sense of mystery. One may prove a theorem via a sequence of unassailable steps, but still with a shallow sense of why it works. The proof might seem to be a shadow of a higher truth, reflecting a deep connection with other apparently disconnected ideas in mathematics. When those connections materialize, the magic can be dazzling.

Of course, once you see how it's done, it should be clear that no actual magic or mystery is involved; all you need to do is summon up the right ingredients, combine them judiciously, avoid getting confused, and the problem solves itself[5]. In short, it bears some resemblance to cooking (so it may be more closely related to Potions after all). But how do you know which ingredients to trot out, and how to combine them, and in what proportion? That, of course, is where all the difficulty, and creativity, lies. And one key to that creativity is abstraction, and a process of uncovering similarities between the problem you're solving and others that are already solved. This, in turn, is aided and abetted by examining the deep structures underlying that which is being investigated.

Regrettably, too many students in high school and even in many university courses, to say nothing of the general public, are deprived of these sensibilities of magic on the one hand, and the sense of depth via abstraction on the other. Instead they are (and I regret to say, for some of us, "we were") dragged through a curriculum that is both dreary and ineffective as preparation for future study. Little if any sense in the *unity* of Math is ever conveyed. Why not *excite* students, rather than burden them with a tedious and uninspiring curriculum which (as so many of us can now testify from experience as teachers) provides an insufficient background for advanced study?

[5] This is somewhat flippant, of course. I am reminded of an alleged quote of Johann Sebastian Bach, on playing keyboard instruments: "There's nothing remarkable about it. All one has to do is hit the right keys at the right time and the instrument plays itself." Easier said than done.

Truth be known, I wish every high school student would read both Arthur Benjamin's book "The Magic of Math" and Eugenia Cheng's "How to Bake π"; and if they haven't read them by college, then they should do it in their first year. Neither are suitable for use as textbooks, but would make great supplementary reading. Both books are enthusiastic, playful, lively, and rich in insight. They are also very different! The subject matter is largely complementary, and they apply quite different strategies to convey their respective points. All the more reason to read them both.

"The Magic of Math" covers a lot of ground. The chapters are all named "The Magic of ______," where Benjamin fills the blank with elements of the set {"Numbers," "Algebra," "9," "Counting," "Fibonacci Numbers," "Proofs," "Geometry," "π," "Trigonometry," "i and e," "Calculus," "Infinity"}. He says his target audience is "anyone who will someday need to take a math course, is currently taking a math course, or is finished taking math courses." I'd say I don't fit any of these categories (I don't think I'll ever *need* to take another math course, I'm not taking one now, but I can never be sure I've finished taking them!), but nevertheless enjoyed it greatly. As is so often the case, anyone can stand to learn from it. It is, like other books roughly in this class that I have looked at over the years, a valuable source of facts that would be useful in introductory (even not-so-introductory) courses for those of us who teach at the university level. I would hope the general reader is not put off by the many calculations, graphs, algebra, and new concepts, because it has the potential to appeal literally to anybody. A significant thread running through the book is a pleasing assortment of mathematical party tricks – the sort of things that a lay audience might well interpret as "magic."

Thus, for example, in "The Magic of Numbers," along with essential basic properties of numbers (including formulas like the one for the sum of the first n natural numbers) we learn a bunch of tricks for mental multiplication and division. In "The Magic of 9," we learn (again mentally) how to figure out the day of the week on which any date falls (most easily between 1901 and 2099). In "Fibonacci," we learn an intriguing trick about the Fibonacci sequence that derives from adding fractions incorrectly and Binet's formula. And in "The Magic of π," how to memorize hundreds of digits of π without actually remembering any actual digits (I always wondered how that was done).

While the tricks are fun and interesting in themselves, the deeper magical aspects of math infuse the entire book. How did $\sqrt{5}$ end up in a formula for the entirely rational (even integral) Fibonacci sequence? How did e, i, π, 1, and 0 all end up in one beautiful relationship? What business do $\sqrt{2}$, e, and π have in a formula for the factorial? Why in the world is the sum of the inverse squares equal to $\pi^2/6$? In deference to these questions, there is also much serious math that is introduced along the way. We encounter modular arithmetic (in "9"), the binomial coefficients and their basic properties (in "Counting"), the Fibonacci numbers, various proof techniques, Euclidean Geometry, why the area of a circle is πr^2, how to determine the height of a mountain without measuring it, and trigonometric functions and their identities.

The book gives one a sense not only of the math, but of how math is done. Benjamin makes it a point to explore, and look for patterns, rather than merely laying out techniques. For example, after exploring basic properties of the Fibonacci numbers, he looks for various patterns in the sequence. By a series of observations, he brings the reader to the conclusion that every k^{th} number in the sequence is a multiple of the k^{th} Fibonacci number (while that may have been borderline intelligible, Benjamin's explanation is far more lucid). Similarly, he helps the reader observe that consecutive pairs of numbers in the sequence are relatively prime, and then, by example, indicates how one might prove it (implicitly leaving it up to the reader to come up with a general proof... perhaps because proofs aren't discussed until the following chapter).

In that next chapter, "Proofs," Benjamin discusses proof by induction, proof by contradiction, direct proof, and nonconstructive proofs. Induction is illustrated by the usual examples of adding up the first n integers, but also by more interesting formulae such as adding the first n cubes, and the first n squares of the Fibonacci numbers. For contradiction we learn that there are irrational number (i.e., $\sqrt{2}$) and why they are irrational. We also learn that there exist irrationals a, b such that a^b is rational, a corollary of the proof that $\sqrt{2}$ is irrational. And, of course, the infinitude of prime numbers. Inductive proofs are not limited here to algebraic or numerical quantities. He also proves inductively how to tile a $2^n \times 2^n$ board with trominoes.

This is by no means an anthology of disconnected facts. Most concepts introduced early in the book are revisited in some way later on, usually multiple times. For example, modular arithmetic is introduced in "9," but it comes up again in "Fibonacci" and "Infinity." While it's not absolutely necessary to read the chapters in strict sequence, it would help in many cases. For example, "Proofs," "Geometry," "π," and "Trigonometry" are all required reading to really appreciate "i and e" (which culminates in the miraculous $e^{i\pi} + 1 = 0$), and "Calculus" and "Infinity" are not very illuminating without having read the chapters that precede it.

Another impressive thing about the book is that it really does real math, and it is remarkable how much of it is based on solid mathematical proof. "Calculus" is an honest introduction to the subject. It gives all the necessary motivation and geometric intuition, including a perfectly rigorous (for its purposes) definition of the derivative, goes over the standard rules and many examples of differentiation. It stops short of integration, although antiderivatives make a cameo appearance in the next chapter. That concluding chapter, "Infinity," dwells primarily on infinite series. I was hoping that the idea of infinity itself would play a more central role there. Cantor's diagonal argument is given a lucid explanation in an "aside," set off from the text, where I think it would have been better placed. However, the chapter features some of the well-known (apparent?) paradoxes regarding infinite series, such as the fact that we may take the sum of the positive integers to equal $-1/12$. The chapter concludes with something of an epilog that has nothing to do with infinity, namely magic squares.

In short, Benjamin leaves few topics in basic mathematics untouched – be they continuous, discrete, combinatorial, algebraic, analytical or geometric. It is all masterfully done.

In "How to Bake π," the math-phobe reader is less likely to be put off by any proliferation of graphs and equations (which is not to say they are absent), and is bound to feel comfortable with its leisurely pace. But that is deceptive. The subtitle, "An Edible Exploration of the Mathematics of Mathematics," refers to its surprising underlying agenda: It is (as far as I know) the first popular account of Category Theory! The "edible" approach is achieved by drawing analogies between baking and mathematics. Each chapter begins with a baking recipe that constitutes a metaphor for the message of that chapter (or part of it, anyway). This makes the book very amusing, brings it to life, and frankly, they look like great recipes. But while these baking analogies are entertaining and illuminating, reader beware: they have the distinct disadvantage of occasionally arousing feelings of hunger.

So how on earth do you introduce a general audience to category theory? Full disclosure: My knowledge of category theory (which I'll sometimes refer to as "CT") is minimal, so in a certain sense I am part of that audience. In fact a key point that the book makes, successfully at that, is that *it can be done*! Now mind you, although you will, in this book, encounter terms like "category," "object," "morphism," "monoid," "group," "universal property," and even (if you look hard enough) "natural transformation," you won't find, and indeed it would be pointless to give, *precise* meanings to these terms. The key to the challenge is to provide the reader with intuition and sufficiently powerful analogies (which, it seems to me, is very much in the spirit of CT anyway). But before getting down to that business, in a Prologue, Cheng enumerate various all-too-common myths of math: that it's only about numbers, that it's all about just getting the right answers, and so forth. The last of these is a question, “How can you do research? – you can’t just discover a new number.” This book is in large part Cheng’s answer to the last question, in terms of the field of her research, which is, as you may guess, category theory.

After that point, the book is divided into two main parts, “Math” and “Category Theory.” Each of these parts is bookended with an *initial* chapter asking a question and a *final* chapter indicating possible answers: “What is Math?” and “What Mathematics Is” for Part 1, and “What is Category Theory?” and “What Category Theory Is” for Part 2. In a way, the book is categorical in structure.

The “Math” part is suitably longer, as it must be to smash those myths and solidly establish in the reader’s mind just what mathematics is, before exploring the mathematics of mathematics. Regarding the “What is Math?” question, it is easier to characterize it in terms of what it’s *like* (roughly speaking, using logic to study systems that obey logical laws), rather than what it truly *is*. Cheng contends that “power and beauty [of mathematics] lie not in the answers it provides or the problems it solves, but in the *light* it sheds.”

We are then taken through a succession of chapters representing core big ideas in mathematics: Abstraction, Principles, Process, Generalization, Internal vs. External, and Axiomatization. As an example of how the baking recipes are used, the recipe for the “Abstraction” chapter is for “mayonnaise or hollandaise sauce.” Here we may think of a recipe as a blueprint, since one can use alternative ingredients in the same recipe. Likewise, you can reason about seemingly different things with the same reasoning, e.g., the symmetries of a triangle and permutations of $\{1, 2, 3\}$.

The “Internal vs. External” chapter is subtle, important, and as it happens, it is where we meet our first commutative diagrams. The chapter begins with a recipe based on ingredients you just happen to have lying around, figuring out what you might do with them. Cheng refers to this as an *internal* motivation. By the same token, one might come up with some mathematical construction and see where it leads, without any specific problem in mind; this too is internally motivated, driven by curiosity. By contrast, external motivation follows from problems that we’re given, where we are driven by a goal. The internal and the external interact very strongly. Considering the example of Fermat’s Last Theorem, Wiles was surely externally motivated by the 300 year old problem. But there was a great deal of internally motivated research into elliptic curves that needed to be worked out before FLT could be proved. This is an important point, as Cheng regards one of CT’s aims as being to find the internal motivation of *everything*; category theory often “bridges the gap between internal and external processes.”

In the “What Mathematics Is” chapter, Cheng asserts that “[it] is a truth universally acknowledged that mathematics is difficult.” As so often happens, this “truth universally acknowledged” is a falsehood: The purpose of mathematics is to make hard problems easy, or at any rate, to distill out what is easy. And furthermore, Cheng contrasts the “easiness” of math with the “hardness” of life: “Math is easy, life is hard, therefore math isn’t life.” Now from my parochial perspective, we have learned all too well from computational complexity that making hard problems easy can be hard! But granted: we are always on the lookout for the easiest solutions, and mathematics is way easier than life by any measure, so I agree.

In Part 2, in between the initial and terminal bookend chapters, we have chapters on Context, Relationships, Structure, Sameness, and Universal Properties. CT is always concerned with *context*: for example, a Möbius strip is trivial in the context of homotopy equivalence, but is very interesting as a vector bundle over a circle. Quadratic equations may or may not have a solution depending on what field you’re working in. CT emphasizes the context in which things are studied, rather than the things themselves. Of all the chapters, “Relationships” *looks* the most like CT. Here is where categories are introduced somewhat formally, and the basic ideas of objects, morphism, composition, etc., are defined here. After the diagrams of this chapter, the purpose of the remainder of the book is not to extend the vocabulary or technicality of the theory, but rather to concentrate more on its meaning.

Thus, baked Alaska, a recipe in which structural integrity is inextricably entangled with process and the effect of the dessert itself, appears as the recipe for the "Structure" chapter, which also draws on parking garages and the dome of St. Paul's cathedral as metaphors. *Universal Properties* can be understood with analogies ranging from a recipe for fruit crisp to the North and South poles, but Cheng doesn't shy away from meaningful mathematical examples, such as the initial and terminal objects in the category of groups (somewhat like multiple North and South poles, all of which correspond). And the final chapter, "What Category Theory Is," takes aim at the deeper meanings, by exploring how the "Trinity of Truth" (which Cheng formulates as *Belief*, *Understanding*, and *Knowledge*) serves to convince mathematicians that something is true.

At times, I could not help feeling that some opportunities were missed to give the reader a more concrete idea of what CT is, by presenting more "real" examples of categories. But at the same time I believe that this is not really the point of the book. And anyway, if, as you read this, you find yourself hungering not for dessert, but for a more technical introduction (and something quicker and more reader-friendly than Mac Lane's "Categories for the Working Mathematician"), I recommend Cheng's excellent article on CT in the "Princeton Companion to Mathematics."

I confess that when I first received this book from the publisher, I only had time to take a cursory glance at it, and took it to be just another popular math book, with gratuitous recipes at the beginning of each chapter. A few months later, I casually picked it up and actually started reading, and quickly realized how very mistaken I was. As I mentioned earlier, CT is not well-known to me, a gap in my mathematical repertoire I have been trying to fill in my spare time over the past couple of years. As I continue to study the literature, I will keep "How to Bake π" at hand for intuition and illumination.